

## AMPLITUDE EQUATIONS FOR A SYSTEM WITH THERMOHALINE CONVECTION

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*The multiple scale expansion method is used to derive amplitude equations for a system with thermohaline convection in the neighborhood of Hopf and Taylor bifurcation points and at the double zero point of the dispersion relation. A complex Ginzburg–Landau equation, a Newell–Whitehead-type equation, and an equation of the  $\varphi^4$  type, respectively, were obtained. Analytic expressions for the coefficients of these equations and their various asymptotic forms are presented. In the case of Hopf bifurcation for low and high frequencies, the amplitude equation reduces to a perturbed nonlinear Schrödinger equation. In the high-frequency limit, structures of the type of “dark” solitons are characteristic of the examined physical system.*

**Introduction.** In the 1980–1990s, a number of papers devoted to the formation of structures in the neighborhood of Hopf bifurcation points for systems with translational invariance along the horizontal appeared in the literature on double-diffusive convection. Oscillations in such systems can give rise to waves of various types (for example, standing, traveling, modulated, and random), which are conveniently studied by constructing amplitude equations [1]. An amplitude equation for a system with convection was first obtained by Newell and Whitehead [2]. It describes two-dimensional thermal convection and has the form of a generalized Ginzburg–Landau equation. Coulett et al. [3] proposed a system of Ginzburg–Landau equations that describes traveling, double-diffusive waves propagating on both sides in a liquid strip which is infinite along the horizontal:

$$\begin{aligned}(\partial_t + s\partial_x)A_R &= (c_0 + ic_1)A_R + (c_2 + ic_3)\partial_x^2 A_R - (c_4 + ic_5)|A_R|^2 A_R - (c_6 + ic_7)|A_L|^2 A_R, \\(\partial_t - s\partial_x)A_L &= (c_0 + ic_1)A_L + (c_2 + ic_3)\partial_x^2 A_L - (c_4 + ic_5)|A_L|^2 A_L - (c_6 + ic_7)|A_R|^2 A_L.\end{aligned}\tag{1}$$

The form of these equations is postulated from general considerations (such as considerations of symmetry); it is assumed that the coefficients in these equations should be derived by asymptotic methods from the partial equations describing a particular physical system.

However, a thorough and well-founded derivation of amplitude equations for double-diffusive systems is not available in the literature there. In many papers, the form of the coefficients in Eqs. (1) is not discussed. In some papers, these coefficients are obtained from various physical considerations. Thus, Cross [4], examining a system with convection for binary mixtures in the limit of low Hopf frequencies, set the coefficients  $c_1$ ,  $c_3$ ,  $c_5$ , and  $c_7$  in Eqs. (1) equal to zero as a first, crude, approximation, motivating this by empirical data indicating an analogy between the case considered and the case of purely temperature convection. Clearly, such assumptions on the form of the coefficients can be rigorously justified only in a rigorous mathematical derivation of amplitude equations.

In papers on double diffusive convection of binary mixtures in bulk and porous media, the Hopf frequency turns out to be unity in the case of oscillatory convection. For thermohaline convection, it is

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reasonable to consider the asymptotic behavior for the Hopf frequency tending to infinity. In this limit, the amplitude equation should become the nonlinear Shroedinger equations governing internal waves in two-dimensional geometry.

In the present paper, using the derivative expansion method, which is a version of the multiple-scale expansion method, we derive amplitude equations for double-diffusive waves in two-dimensional, horizontally infinite geometry in the neighborhood of the Hopf and Taylor bifurcation points and the double zero point of the dispersion relation. Idealized boundary conditions are used. In the case of Hopf bifurcation, the amplitude equation for waves propagating only in one direction is examined. Analytic expressions for the coefficients of these equations are obtained. Their various asymptotic forms are studied, and asymptotic forms of the amplitude equations for various parameter values are discussed.

**1. Formulation of the Problem; Basic Equations.** The initial equations describe two-dimensional thermohaline convection in a liquid layer of thickness  $h$  bounded by two infinite, plane, horizontal boundaries. The liquid moves in a vertical plane and the motion is described by the stream function  $\psi(t, x, z)$ . The horizontal  $x$  and vertical  $z$  space variables are used; the time is denoted by  $t$ . It is assumed that there are no distributed sources of heat and salt, and on the upper and lower boundaries of the regions, these quantities have constant values. Hence, the main distribution of temperature and salinity is linear along the vertical and does not depend on time. The variables  $\theta(t, x, z)$  and  $\xi(t, x, z)$  describe variations in the temperature and salinity about this main distribution. There are two types of thermohaline convection: the finger type, in which the warmer and more saline liquid is at the upper boundary of the regions, and the diffusive type, in which the temperature and salinity are greater at the lower boundary. In the present paper, we study the second type.

The evolution equations in the Boussinesq approximation in dimensionless form are a system of nonlinear equations in first-order partial derivatives with respect to time that depend on four parameters: the Prandtl number  $\sigma$ , the Lewis number  $\tau$  ( $0 < \tau < 1$ ), and the temperature  $R_T$  and salinity  $R_S$  Rayleigh numbers [5, 6]:

$$(\partial_t - \sigma\Delta)\Delta\psi + \sigma(R_S\partial_x\xi - R_T\partial_x\theta) = -J(\psi, \Delta\psi), \quad (2)$$

$$(\partial_t - \Delta)\theta - \partial_x\psi = -J(\psi, \theta), \quad (\partial_t - \tau\Delta)\xi - \partial_x\psi = -J(\psi, \xi).$$

Here we have introduced the Jacobian  $J(f, g) = \partial_x f \partial_z g - \partial_x g \partial_z f$ . The boundary conditions for the dependent variables are chosen to be zero, which implies that the temperature and salinity at the boundaries of the region are constant, the vortex vanishes at the boundaries, and the boundaries are impermeable:

$$\psi = \partial_z^2\psi = \theta = \xi = 0 \quad \text{at} \quad z = 0, 1. \quad (3)$$

In the literature, these boundary conditions are usually called free-slip conditions or simply free conditions since the horizontal velocity component at the boundary does not vanish.

The space variables are made nondimensional with respect to the thickness of the liquid layer  $h$ . As the time scale, we use the quantity  $t_0 = h^2/\chi$ , where  $\chi$  is the thermal diffusivity of the liquid. The vertical and horizontal components of the liquid-velocity field are defined by the formulas

$$v_z = \frac{\chi}{h} \partial_x\psi, \quad v_x = -\frac{\chi}{h} \partial_z\psi.$$

The dimensional temperature  $T$  and salinity  $S$  are given by the relations

$$T(t, x, z) = T_- + \delta T[1 - z + \theta(t, x, z)], \quad S(t, x, z) = S_- + \delta S[1 - z + \xi(t, x, z)].$$

Here  $\delta T = T_+ - T_-$ ,  $\delta S = S_+ - S_-$ , and  $T_+$ ,  $T_-$  and  $S_+$ ,  $S_-$  are the temperatures and salinities on the lower and upper boundaries of the region, respectively. The temperature and salinity Rayleigh numbers can be expressed in terms of the parameters of the problem:

$$R_T = \frac{g\alpha h^3}{\chi\nu} \delta T, \quad R_S = \frac{g\gamma h^3}{\chi\nu} \delta S.$$

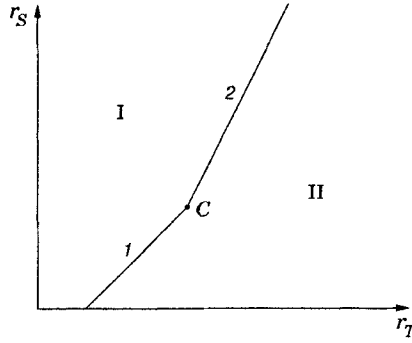


Fig. 1

Here  $g$  is the acceleration of gravity,  $\nu$  is the viscosity of the liquid, and  $\alpha$  and  $\gamma$  are the temperature and salinity coefficient of cubic expansions.

**2. Dispersion Relation and Its Consequences.** We consider a system of partial differential equations that is derived by linearization of the initial system (2) in the neighborhood of the trivial solution:

$$(\partial_t - \sigma\Delta)\Delta\psi + \sigma(R_S\partial_x\xi - R_T\partial_x\theta) = 0, \quad (\partial_t - \Delta)\theta - \partial_x\psi = 0, \quad (\partial_t - \tau\Delta)\xi - \partial_x\psi = 0. \quad (4)$$

These equations are solved subject to boundary conditions (3) by the method of separation of variables. We seek a solution in the form

$$\varphi = [\mathbf{A}_1 \exp(\lambda t - i\beta x) + \bar{\mathbf{A}}_1 \exp(\bar{\lambda} t + i\beta x)] \sin(\pi z). \quad (5)$$

Here the bar denotes complex conjugation,  $\varphi = (\psi, \theta, \xi)$  is the vector of the basic dependent quantities,  $\beta$  is the horizontal wavenumber, and  $\mathbf{A}_1 = (a_{A1}, a_{T1}, a_{S1})$  is the amplitude vector. For  $a_{A1}$ , we use the notation  $A \equiv a_{A1}$ .

Substitution of (5) into (4) gives a system of algebraic equations for the variables  $a_{A1}$ ,  $a_{T1}$ , and  $a_{S1}$ . The condition for the existence of solutions of this system takes the form of an algebraic equation of the third order in  $\lambda$  [6]:

$$\lambda^3 + (1 + \tau + \sigma)k^2\lambda^2 + [(\tau + \sigma + \tau\sigma) + \sigma(r_S - r_T)]k^4\lambda + \sigma(r_S - \tau r_T + \tau)k^6 = 0. \quad (6)$$

Here we introduced the wavenumber  $k^2 = \pi^2 + \beta^2$ , and the normalized Rayleigh numbers  $r_T = R_T/R^*$  and  $r_S = R_S/R^*$ , where  $R^* = k^4(k/\beta)^2$  is the Rayleigh number, for which there is loss of stability of the steady state for purely temperature convection.

Equation (6) has three roots, two of which can be complex conjugate. In the physical system considered, loss of stability occurs when with change in the bifurcation parameters  $r_T$  and  $r_S$ , one or several roots pass through zero or gain a positive real part if they are complex.

In the plane of the parameters  $r_T$  and  $r_S$  (see Fig. 1), it is possible to distinguish regions I and II, on whose boundary there is loss of stability. The boundary itself consists of two rectilinear segments. On segment 1, Taylor bifurcation is observed when one of the roots of the dispersion relation passes through zero, which gives rise to steady drum-type convection. On segment 2, Hopf bifurcation takes place when the real parts of two complex conjugate roots become positive. As a result, oscillatory convection occurs. The segments adjoin at the point  $C$ , at which the dispersion relation (6) has a double root. At this point, the parameter values are defined by

$$r_{T1} = \frac{1}{\sigma} \frac{\tau + \sigma}{1 - \tau}, \quad r_{S1} = \frac{\tau^2}{\sigma} \frac{1 + \sigma}{1 - \tau}.$$

The straight lines on which Taylor and Hopf bifurcations are observed, are given, respectively, by the equations

$$r_T = \frac{1}{\tau} r_S + 1, \quad r_T = 1 + \frac{\tau}{\sigma} (1 + \tau + \sigma) + \frac{\tau + \sigma}{1 + \sigma} r_S.$$

The oscillation frequency of oscillatory convection is determined by the imaginary part  $\lambda$  and is expressed in terms of the reduced frequency  $\Omega$  as  $\text{Im}(\lambda) = \Omega k^2$ , and  $\Omega$  is, in turn, calculated from the formula

$$\Omega^2 = -\tau^2 + \frac{1-\tau}{1+\sigma} \sigma r_S \quad (\lambda = i\Omega k^2).$$

Below, the reduced frequency  $\Omega$  is called the Hopf frequency.

**3. Slow Variables and Expansion of the Solutions.** We consider the equations of double-diffusive convection in the neighborhood of a certain bifurcation point, for which the temperature and salinity Rayleigh numbers are denoted by  $R_T^*$  and  $R_S^*$ , respectively. In this case, the Rayleigh number is written as

$$R_T = R_T^*(1 + \varepsilon^2 \eta), \quad R_S = R_S^*(1 + \varepsilon^2 \eta_S).$$

The values of  $\eta$  and  $\eta_S$  are of the order of unity, and the small parameter  $\varepsilon$  shows how far from the bifurcation point the examined system is located. To derive amplitude equations, we use the derivative expansion method of [7, 8]. We introduce the slow variables

$$T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \quad X_1 = \varepsilon x.$$

Next, into the basic equations (2), we introduce the extended derivative by the rules

$$\partial_t \rightarrow \partial_t + \varepsilon \partial_{T_1} + \varepsilon^2 \partial_{T_2}, \quad \partial_x \rightarrow \partial_x + \varepsilon \partial_{X_1}. \quad (7)$$

The dependent variables are represented as series in the small parameter:

$$\varphi = \sum_{n=1}^3 \varepsilon^n \varphi_n(x, z, t, X_1, T_1, T_2) + O(\varepsilon^4).$$

Substituting these expressions into Eqs. (2) with derivatives extended according to (7) and grouping terms with the same power of  $\varepsilon$ , we obtain

$$O(\varepsilon): \quad L^* \varphi_1 = 0,$$

$$O(\varepsilon^2): \quad L^* \varphi_2 = -(L_1 \partial_{T_1} - L_2 \partial_{X_1}) \varphi_1 - M_1 \varphi_1,$$

$$O(\varepsilon^3): \quad L^* \varphi_3 = -(L_1 \partial_{T_1} - L_2 \partial_{X_1}) \varphi_2 - (L_1 \partial_{T_2} + L_3 \partial_{X_1}^2 + L_4 \partial_{X_1} \partial_{T_1} + L_5) \varphi_1 - M_2(\varphi_1, \varphi_2).$$

Here the operators  $L_1$ ,  $L_3$ , and  $L_4$  have diagonal form:  $\text{diag } L_1 = (\Delta, 1, 1)$ ,  $\text{diag } L_3 = (\partial_t - 2\sigma\Delta - 4\sigma\partial_x^2, -1, -1)$ , and  $\text{diag } L_4 = (2\partial_x, 0, 0)$ ; the operators  $L^*$  and  $L_2$  can be written as

$$L^* = \begin{pmatrix} (\partial_t - \sigma\Delta)\Delta & -\sigma R_T^* \partial_x & \sigma R_S^* \partial_x \\ -\partial_x & \partial_t - \Delta & 0 \\ -\partial_x & 0 & \partial_t - \tau\Delta \end{pmatrix}, \quad L_2 = \begin{pmatrix} -2(\partial_t - 2\sigma\Delta)\partial_x & \sigma R_T^* & -\sigma R_S^* \\ 1 & 2\partial_x & 0 \\ 1 & 0 & 2\tau\partial_x \end{pmatrix}.$$

In the operator  $L_5$ , only the upper row is different from zero:  $(0, -\sigma R_T^* \eta \partial_x, \sigma R_S^* \eta_S \partial_x)$ . The vectors  $M_i = (M_{Ai}, M_{Ti}, M_{Si})$  with nonlinear terms have the components

$$M_{A1} = J(\psi_1, \Delta\psi_1), \quad M_{T1} = J(\psi_1, \theta_1), \quad M_{S1} = J(\psi_1, \xi_1),$$

$$M_{A2} = J(\psi_2, \Delta\psi_1) + J(\psi_1, \Delta\psi_2) + J(\psi_1, 2\partial_x \partial_{X_1} \psi_1) + \partial_z \Delta\psi_1 \partial_{X_1} \psi_1 - \partial_z \psi_1 \partial_{X_1} \Delta\psi_1,$$

$$M_{T2} = J(\psi_1, \theta_2) + J(\psi_2, \theta_1) + \partial_z \theta_1 \partial_{X_1} \psi_1 - \partial_z \psi_1 \partial_{X_1} \theta_1,$$

$$M_{S2} = J(\psi_1, \xi_2) + J(\psi_2, \xi_1) + \partial_z \xi_1 \partial_{X_1} \psi_1 - \partial_z \psi_1 \partial_{X_1} \xi_1.$$

The systems obtained can be written in general form:

$$L^* \varphi_i = Q'_i + P_i.$$

Here the functions  $Q'_i$  consist of terms that are in resonance with the left side of the equations. The condition of the absence of secular terms in solutions of similar systems of equations is known (see [7, 8]) to reduce to

the requirement that the functions  $Q'_i$  and the solutions of the conjugate homogeneous equation  $(L^*)^\times \varphi_i^\times = 0$  be orthogonal. We now derive the relation to which the condition of the absence of secular terms reduces in this case and which will be used below to derive amplitude equations. Let us consider the inhomogeneous system of algebraic equations that is obtained from (4) by choosing the single-mode form (5) and substituting functions  $Q_i = (Q_{Ai}, Q_{Ti}, Q_{Si})$  such that  $Q'_i = Q_i \exp(\lambda t - i\beta x) + \bar{Q}_i \exp(\bar{\lambda} t + i\beta x)$  into the right side of the homogeneous system:

$$\begin{aligned} (\lambda + \sigma k^2)(-k^2)a_{Ai} + \sigma R_T^* i\beta a_{Ti} - \sigma R_S^* i\beta a_{Si} &= Q_{Ai}, \\ (\lambda + k^2)a_{Ti} + i\beta a_{Ai} &= Q_{Ti}, \quad (\lambda + \tau k^2)a_{Si} + i\beta a_{Ai} = Q_{Si}. \end{aligned} \quad (8)$$

The solvability condition for this system of equations is formulated as the requirement that the right side be orthogonal to the solution of the conjugate homogeneous system [9] [ $1. -i\beta\sigma R_T^*/(\lambda+k^2)$  and  $i\beta\sigma R_S^*/(\lambda+\tau k^2)$ ] and reduces to the equation

$$(\lambda + k^2)k^6\sigma r_S^* Q_{Si} - (\lambda + \tau k^2)k^6\sigma r_T^* Q_{Ti} - (\lambda + k^2)(\lambda + \tau k^2)i\beta Q_{Ai} = 0. \quad (9)$$

For  $\lambda = 0$ , this relation is simplified:

$$\frac{1}{\tau} r_S^* Q_{Si} - r_T^* Q_{Ti} - \frac{i\beta}{\sigma k^4} Q_{Ai} = 0.$$

**4. Derivation of Amplitude Equations.** Let us assume that the solution of the equations for  $\varphi_1$  has the form (5) and the amplitude of this solution now depends on the slow variables:  $A = A(T_1, X_1, T_2)$ . Substituting this solution into the equations for  $\varphi_2$ , we obtain a system of equations of the form (8) in which the functions  $Q_2$  are written as

$$\begin{aligned} Q_{A2} &= k^2 \partial_{T_1} A + i\beta\sigma \left( \frac{R_S^*}{\lambda + \tau k^2} - \frac{R_T^*}{\lambda + k^2} + 4k^2 + \frac{2\lambda}{\sigma} \right) \partial_{X_1} A, \\ Q_{T2} &= \frac{i\beta}{\lambda + k^2} \partial_{T_1} A + \left( 1 - \frac{2\beta^2}{\lambda + k^2} \right) \partial_{X_1} A, \quad Q_{S2} = \frac{i\beta}{\lambda + \tau k^2} \partial_{T_1} A + \left( 1 - \frac{2\tau\beta^2}{\lambda + \tau k^2} \right) \partial_{X_1} A. \end{aligned}$$

For this algebraic system to be solvable, it is necessary that condition (9) be satisfied. At different bifurcation points, this condition is formulated as different equations. We consider successively the equations obtained from the solvability condition of the indicated system at the bifurcation points characteristic of the physical system considered.

In the last relations, we substitute the value of  $\lambda$  at the Hopf bifurcation point  $\lambda = i\Omega k^2$  and set  $k^2/\beta^2 = 3$  and  $\beta = \pi/\sqrt{2}$ , which is valid for the oscillation mode that is the first to lose stability [5]. In addition, we take into account the relations

$$r_T^* = \frac{1}{\sigma} \frac{\sigma + \tau}{1 - \tau} (\Omega^2 + 1), \quad r_S^* = \frac{1}{\sigma} \frac{\sigma + 1}{1 - \tau} (\Omega^2 + \tau^2).$$

Then, Eq. (9) can be written as

$$\partial_{T_1} A + \sqrt{2}\pi\Omega \partial_{X_1} A = 0$$

and solved in general form by introducing the new slow variable  $X = X_1 - \sqrt{2}\pi\Omega T_1$ . If we assume that the amplitude  $A(X, T_2)$  depends on  $X_1$  and  $T_1$  only via  $X$ , this equation becomes an identity.

In the other cases, where we consider the system at the Taylor bifurcation point or at the double zero point, the solvability condition (9) has the form

$$\frac{1}{\tau} (1 - \tau) \left( r_T^* - \frac{\sigma + \tau}{\sigma(1 - \tau)} \right) \partial_{T_1} A + 2i\beta \left( \frac{k^2}{\beta^2} - 3 \right) \partial_{X_1} A = 0. \quad (10)$$

If in this equation, as above,  $k^2/\beta^2 = 3$ , i.e., the least stable oscillation mode is considered, then  $\partial_{T_1} A = 0$  holds for the case of Taylor bifurcation. In the case of the double zero point, Eq. (10) is satisfied identically.

**5. Amplitude Equation at the Hopf Bifurcation Point.** We now write the solution for  $\varphi_2$  with the wavenumber for which there is loss of stability of the steady state:

$$\varphi_2 = [A_2 \exp(i\Omega k^2 t - i\beta x) + \bar{A}_2 \exp(-i\Omega k^2 t + i\beta x)] \sin(\pi z) + B_2 \sin(2\pi z).$$

Here  $A_2 = (a_{A2}, a_{T2}, a_{S2})$  and  $B_2 = (0, b_{T2}, b_{S2})$  are vectors that depend on the slow variables. The components of these vectors have the values

$$b_{T2} = -\frac{1}{6\pi} \frac{|A|^2}{1 + \Omega^2}, \quad b_{S2} = -\frac{1}{6\pi} \frac{|A|^2}{\tau^2 + \Omega^2},$$

$$a_{T2} = \frac{2}{9\pi^2} \frac{1}{1 + i\Omega} \left( \partial_X A - \frac{3i\pi}{\sqrt{2}} a_{A2} \right), \quad a_{S2} = \frac{2}{9\pi^2} \frac{1}{\tau + i\Omega} \left( \partial_X A - \frac{3i\pi}{\sqrt{2}} a_{A2} \right).$$

Using the solutions given above, we formulate a system of equations from which it is possible to find  $\varphi_3$ . This system of equations, as the system for  $\varphi_2$ , has the form (8). Then, we can write the functions  $Q_3$  as follows, retaining in them only terms with  $A(X, T_2)$ :

$$Q_{A3} = \frac{3}{2} \pi^2 \left\{ \partial_{T_2} A - \frac{1}{3} (4i\Omega + 7\sigma) \partial_X^2 A + \frac{3\pi^2}{2(1-\tau)} [(\sigma+1)(\tau-i\Omega)\eta_S - (\sigma+\tau)(1+i\Omega)\eta] A \right\},$$

$$Q_{T3} = \frac{i\sqrt{2}}{3\pi} \frac{1}{1+i\Omega} \left[ \partial_{T_2} A - \frac{1}{3} (2i\Omega + 5) \partial_X^2 A + \frac{\pi^2}{4} \frac{1}{1-i\Omega} A |A|^2 \right],$$

$$Q_{S3} = \frac{i\sqrt{2}}{3\pi} \frac{1}{\tau+i\Omega} \left[ \partial_{T_2} A - \frac{1}{3} (2i\Omega + 5\tau) \partial_X^2 A + \frac{\pi^2}{4} \frac{1}{\tau-i\Omega} A |A|^2 \right].$$

Condition (9) for system (8) has the form

$$(\sigma+1)(\tau-i\Omega)Q_{S3} - (\sigma+\tau)(1-i\Omega)Q_{T3} - (1-\tau)(i\beta/k^4)Q_{A3} = 0.$$

After transformations, we find that the amplitude  $A(X, T_2)$  should satisfy the complex Ginzburg-Landau equation

$$\partial_{T_2} A = \alpha_1 A + \beta_1 A |A|^2 + \gamma_1 \partial_X^2 A. \quad (11)$$

The coefficients in this equation are defined by the formulas

$$\alpha_1 = \frac{3i\pi^2 [\eta_S(\sigma+1)(\Omega^2 + \tau^2)(i\Omega + 1) - \eta(\sigma+\tau)(\Omega^2 + 1)(i\Omega + \tau)]}{4\Omega[i\Omega + (1 + \sigma + \tau)](1-\tau)},$$

$$\beta_1 = -\frac{i\pi^2}{8\Omega}, \quad \gamma_1 = i\Omega + 2 \frac{(\sigma + \sigma\tau + \tau)\Omega - i\sigma\tau}{\Omega[i\Omega + (1 + \sigma + \tau)]}.$$

**6. Equation in the Form of a Perturbed Nonlinear Shroedinger Equation.** For further investigation, the equation obtained can be brought to a more convenient form. We set  $\eta_S = 0$ . This implies that the behavior of the system can be controlled by changing the temperature gradient in the layer, while the salinity gradient remains constant and equal to the critical value. The coefficient  $\alpha_R$  ( $i\alpha_1/\eta = \alpha_R + i\alpha_I$ ) is eliminated from the equation by changing the dependent variable by the formula  $A = A' \exp(-i\alpha_R \eta T_2)$ . Equation (11) then becomes

$$i\partial_{T_2} A' + \gamma_R \partial_X^2 A' - \beta_R A' |A'|^2 = i\alpha_I \eta A' + i\gamma_I \partial_X^2 A'. \quad (12)$$

Here

$$\alpha_R = \frac{3}{4} \pi^2 \frac{\sigma + \tau}{1 - \tau} \frac{\Omega^2 + 1}{\Omega^2 + (1 + \tau + \sigma)^2} \left( \Omega + \frac{\tau(1 + \tau + \sigma)}{\Omega} \right),$$

$$\alpha_I = \frac{3}{4} \pi^2 \frac{(\sigma + \tau)(\sigma + 1)}{1 - \tau} \frac{\Omega^2 + 1}{\Omega^2 + (1 + \tau + \sigma)^2}, \quad \beta_R = \frac{\pi^2}{8\Omega},$$

$$\gamma_R = \Omega - 2 \frac{(\sigma + \sigma\tau + \tau)\Omega^2 + \sigma\tau(1 + \tau + \sigma)}{\Omega[\Omega^2 + (1 + \tau + \sigma)^2]}, \quad \gamma_I = 2 \frac{(\sigma + \tau)(1 + \tau + \sigma + \tau\sigma)}{\Omega^2 + (1 + \tau + \sigma)^2}.$$

Thus, the amplitude equation becomes a nonlinear Shroedinger equation (NSE) with perturbation on the right side. We note that the coefficient values in this equation do not coincide with those given by Bretherton and Spiegel [10], who studied an equation of the type (11) obtained by the method of expansion of a linear dispersion relation in the neighborhood of a critical wavenumber.

It is known [11] that for a dissipatively perturbed NSE, perturbation changes the form of solutions only slightly. As a result, as the perturbing terms tend to zero, these solutions become solutions of an NSE with no perturbed right side, which can be solved in general form by the method of the inverse dissipation problem [7]. If an NSE has soliton solutions (both envelope solitons and solitons above a field of finite density), Eq. (12) has solutions of the same form with rather small perturbing terms. The type of NSE is determined by the sign of the second derivative. In this case, the sign of the coefficient  $\gamma_R$  varies from zero to infinity with change in  $\Omega$ . Hence, in the problem considered, two types of NSE are possible:

— for  $\Omega \rightarrow 0$ ,

$$\gamma_R = -\frac{2\sigma\tau}{1+\tau+\sigma}\Omega^{-1} + \left(1 - \frac{2(\tau+\sigma+\tau\sigma)}{(1+\tau+\sigma)^2} + \frac{2\tau\sigma}{(1+\tau+\sigma)^3}\right)\Omega + O(\Omega^3),$$

$$\gamma_I = 2 + \frac{2}{1+\tau+\sigma}\left(\tau\sigma - 1 - \frac{\tau\sigma}{1+\tau+\sigma}\right) + O(\Omega^2);$$

— for  $\Omega \rightarrow \infty$ ,

$$\gamma_R = \Omega - 2(\tau+\sigma+\tau\sigma)\Omega^{-1} + O(\Omega^{-3}), \quad \gamma_I = 2(\tau+\sigma)(1+\tau+\sigma+\tau\sigma)\Omega^{-2} + O(\Omega^{-4}).$$

In the limit  $\Omega = 0$ , the coefficient  $\gamma_R$  becomes infinity and Eq. (12) loses meaning. This limiting case corresponds to the double zero point of the dispersion relation. The amplitude equation in the  $\varepsilon^2$ -neighborhood of this point will be deduced below. As  $\Omega$  increases from zero to infinity,  $\gamma_R$  changes sign, whereas  $\gamma_I$  decreases monotonically, remaining always positive. The frequency  $\Omega_0$  at which  $\gamma_R$  vanishes is determined from the formula

$$\Omega_0^2 \doteq \frac{1}{2}(1+\sigma^2+\tau^2)\left(\sqrt{1 + \frac{8\sigma\tau(1+\tau+\sigma)}{(1+\sigma^2+\tau^2)^2}} - 1\right).$$

For rather large  $\sigma$  or small  $\tau$ , this formula has the asymptotic form  $\Omega_0^2 \approx 2\tau\sigma(1+\tau+\sigma)/(1+\tau^2+\sigma^2)$ .

**7. Transformation to a Nonlinear Shroedinger Equation.** We consider two cases where the amplitude equation derived above becomes an NSE. Using the substitution

$$A = \sqrt{\alpha_I/\beta_R} \exp[-i(\alpha_R + \alpha_I\rho^2)T_2]F(\alpha_I T_2, \sqrt{\alpha_I/\gamma_R}X),$$

where  $\rho$  is a positive constant, we bring Eq. (12) to the form

$$iF_T + F_{XX} - F(|F|^2 - \rho^2) = i\eta F + i\mu F_{XX}, \quad (13)$$

where  $\mu = \gamma_I/\gamma_R$ . Here and below, the subscripts  $T$  and  $X$  denote partial derivatives with respect to the slow time  $T_2$  and the  $X$  coordinate, respectively. The coefficient  $\mu$  tends to zero with increase in  $\Omega$  according to the asymptotic relation  $\mu \approx 2(\tau+\sigma)(1+\tau+\sigma+\tau\sigma)\Omega^{-3}$ . In addition, in the immediate vicinity of the Hopf bifurcation point (in the  $\varepsilon^3$  neighborhood), the first term on the right side of Eq. (13) can be eliminated. The second term can also be ignored if the frequency  $\Omega$  is sufficiently high. As a result, Eq. (12) becomes the NSE

$$iF_T + F_{XX} - F(|F|^2 - \rho^2) = 0.$$

This equation has solutions that are known as solitons of finite density or “dark” solitons [12]:

$$F = \rho \frac{\exp(i\zeta) + \exp \Phi}{1 + \exp \Phi}, \quad |F|^2 = \rho^2 \left(1 - \frac{\sin^2(\zeta/2)}{\cosh^2 \Phi}\right), \quad (14)$$

$$\Phi = -\rho T \sin \zeta \pm (X - X_0) \sqrt{2\rho} \sin(\zeta/2).$$

The parameters  $\zeta$  and  $X_0$  characterize the width and initial position of the soliton, respectively.

Thus, the present investigation shows that for the physical system considered, along with other solutions, there can be solutions of the type of “dark” solitons, and this is true in the limit of high Hopf frequencies. Apparently, double-diffusive convection at high Hopf frequencies can occur in ocean systems. An example of these systems is a so-called *thermohaline staircase* [13]. Inversions of a thermohaline staircase often have stratification parameters, which correspond to the beginning of diffusive convection, and the Hopf frequency  $\Omega$  is of the order of  $10^3$ – $10^5$ .

When the Hopf frequency tends to zero, Eq. (12) takes a different asymptotic form. In this case, we set

$$A = \sqrt{\alpha_I/\beta_R} \exp(-i\alpha_R T_2) F(\alpha_I T_2, \sqrt{-\alpha_I/\gamma_R X}).$$

Then,

$$iF_T - F_{XX} - F|F|^2 = i\eta F + i\mu F_{XX},$$

where  $\mu$  has the following low-frequency asymptotic form:

$$\mu \approx -\Omega \left( 1 + \frac{\tau + \sigma}{\tau\sigma} - \frac{1}{1 + \tau + \sigma} \right).$$

Thus,  $\mu \rightarrow 0$  as  $\Omega \rightarrow 0$ . As in the previous case, the first term on the right side of the equation can be eliminated by assuming that the system is in the immediate vicinity (in the  $\varepsilon^3$  neighborhood) of the Hopf bifurcation point. Then, again, Eq. (12) takes the form of an NSE:

$$iF_T = F_{XX} + F|F|^2.$$

This equation has well-known solutions in the form of envelope solitons.

It is interesting that localized wave packets, with which soliton solution can be compared, were observed in experiments on convection of binary mixtures at rather low Hopf frequencies (see, for example, [14, 15]).

**8. Equations at the Taylor Bifurcation Points and Double Zero Point.** We consider the case of Taylor bifurcation or bifurcation to steady drum-type convection. On the straight line on which this bifurcation occurs, the dispersion relation has a first-order root. For terms of the order of  $O(\varepsilon^2)$ , the equation has the form  $\partial_{T_1} A = 0$ , i.e., the amplitude does not depend on the slow variable  $T_1$ . For terms of the order of  $O(\varepsilon^3)$  of the functions  $Q_3$ , we obtain the expressions

$$Q_{A3} = \frac{9}{4} \sigma \pi^4 [r_T^*(\eta_S - \eta) - \eta_S] A + \frac{3\pi}{\sqrt{2}} \partial_{T_2} A - \frac{7\sigma\pi^2}{2} \partial_{X_1}^2 A,$$

$$Q_{T3} = \frac{i\pi}{6\sqrt{2}} \left( A|A|^2 + \frac{4}{\pi^2} \partial_{T_2} A - \frac{20}{3\pi^2} \partial_{X_1}^2 A \right), \quad Q_{S3} = \frac{i\pi}{6\tau^2\sqrt{2}} \left( A|A|^2 + \frac{4\tau}{\pi^2} \partial_{T_2} A - \frac{20\tau^2}{3\pi^2} \partial_{X_1}^2 A \right).$$

Substituting these formulas into the compatibility condition, we have the amplitude equation

$$\partial_{T_2} A = \alpha_3 A - \beta_3 A|A|^2 - \gamma_3 \partial_{X_1}^2 A, \quad (15)$$

where

$$\alpha_3 = \frac{3}{2} \pi^2 \tau \frac{r_T^*(\eta_S - \eta) - \eta_S}{r_T^*(1 - \tau) - (1 + \tau/\sigma)}, \quad \beta_3 = \frac{\pi^2}{4\tau} \frac{r_T^*(1 - \tau^2) - 1}{r_T^*(1 - \tau) - (1 + \tau/\sigma)},$$

$$\gamma_3 = \frac{4\tau}{r_T^*(1 - \tau) - (1 + \tau/\sigma)}.$$

This equation is similar in form to the equation derived in [2] and reduces to it if a salinity gradient is absent.

We consider the  $\varepsilon^2$  neighborhood of the point of intersection of the straight lines on which Hopf and Taylor bifurcations are observed. At this point, the dispersion relation has a second-order root (Takens-Bogdanov bifurcation). As noted above, for the case of the most unstable convective mode, the equation obtained for terms of the order of  $O(\varepsilon^2)$  is satisfied identically. Therefore, it is not necessary to use the



variable  $T_2$  or to introduce other slow variables. For terms of the order of  $O(\varepsilon^3)$  of the functions  $Q_3$ , we obtain the expressions

$$Q_{A3} = \frac{9\pi^4}{4(1-\tau)} [(\sigma+1)\eta_S - (1+\sigma/\tau)\eta]A - \frac{i\pi}{\sqrt{2}} \partial_{X_1} \partial_{T_1} A - \frac{7\sigma\pi^2}{2} \partial_{X_1}^2 A,$$

$$Q_{T3} = \frac{i\pi}{6\sqrt{2}} \left( A|A|^2 - \frac{8}{3\pi^4} \partial_{T_1}^2 A - \frac{20}{3\pi^2} \partial_{X_1}^2 A \right) + \frac{2}{9\pi^2} \partial_{X_1} \partial_{T_1} A,$$

$$Q_{S3} = \frac{i\pi}{6\tau^2\sqrt{2}} \left( A|A|^2 - \frac{8}{3\pi^4} \partial_{T_1}^2 A - \frac{20\tau}{3\pi^2} \partial_{X_1}^2 A \right) + \frac{2}{9\pi^2\tau} \partial_{X_1} \partial_{T_1} A.$$

After substitution of these expressions into the condition of the absence of secular terms, we obtain the equation

$$\partial_{T_1}^2 A - c^2 \partial_{X_1}^2 A = \alpha_2 A + \beta_2 A|A|^2, \quad (16)$$

where

$$c^2 = \frac{6\pi^2\sigma\tau}{1+\tau+\sigma}, \quad \beta_2 = \frac{3}{8}\pi^4, \quad \alpha_2 = \frac{9}{4}\pi^4\tau^2 \frac{(1+\sigma/\tau)\eta - (1+\sigma)\eta_S}{(1-\tau)(1+\tau+\sigma)}.$$

Equations of this type are known as  $\varphi^4$ -equations, and they cannot be integrated accurately by the method of the inverse dissipation problem [7]. Some papers [16–18] consider amplitude equations at the double point for the convection of binary mixtures. According to [19], the results obtained for thermohaline convection are extended to the case of convection of binary mixtures, where it is necessary to allow for the thermodiffusion effect. Therefore, for the last case, all the equations at bifurcation points derived in the present paper are valid with the parameters of the problem converted accordingly (Prandtl, Lewis, and Rayleigh numbers). Knobloch [18] obtained an amplitude equation at the double zero point that has the form  $\partial_{T_1}^2 A = C_1 A + C_2 A|A|^2$  in the main order in  $\varepsilon$  ( $C_1$  and  $C_2$  are constants). Equation (16) can be regarded as its extension to the case of spatial modulations. Brand et al. [16] gives another amplitude equation at the double zero point, which includes a term with a third derivative of the form  $\partial_t \partial_x^2 A$ . Therefore, it differs from the equations derived by the multiple-scale expansion method used in the present paper.

**9. Conclusions.** 1. The derivative expansion method is used to derive amplitude equations for a system with thermohaline convection in the neighborhood of the main bifurcation points characteristic of this system. In particular, within the framework of a unified approach, we obtained the complex Ginzburg–Landau equation (11) in the case of Hopf bifurcation, the Newell–Whitehead equation (15) in the case of Taylor bifurcation, and Eq. (16) of the  $\varphi^4$  type in the neighborhood of the double zero point of the dispersion relation.

2. Analytic expressions for the coefficients of the equations considered are given. For the equation in the neighborhood of the Hopf bifurcation points, the formulas specifying its coefficients refine the previous results of [10]. For the other two equations, such formulas, as far as we know, have not been previously reported in the literature.

3. It is shown that, for low and high frequencies, the amplitude equation in the neighborhood of the Hopf bifurcation points reduces to the perturbed nonlinear Shroedinger equations (12) with characteristic solutions in the form of envelope solitons. In the high-frequency limit, the type of “dark” solitons (14) are characteristic of the examined physical system.

4. The equation of the type of  $\varphi^4$  derived at the double zero point of the dispersion relation can be regarded as an extension of the equation obtained in [18] to the case of slow spatial modulations of the amplitude.

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